Towards a Database of Belyĭ Maps

Myles Ashitey (Pomona College), Brian Bishop (Pomona College), Kendall Bowens (Tuskegee University),
Tayler Fernandes Nuñez (Northeastern University), Kayla Gibson (University of Iowa), and Cameron Thomas (Morehouse College)

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Abstract

A Belyĭ map $\beta : \mathbb{P}^1(\mathbb{C}) \to : //www.overleaf.com/1488592572mbcfpctpvcqmathb$ is a rational function with at most three critical values; we may assume these are $\{0,\,1,\,\infty\}$. A Dessin d'Enfant is a planar bipartite graph on the sphere obtained by considering the preimage of a path between two of these critical values, usually taken to be the line segment from 0 to 1. Such graphs can be drawn on the sphere by composing with stereographic projection: $\beta^{-1}([0,1]) \subseteq \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$. This project sought to either create or expand on a database of such Belyĭ pairs, their corresponding Dessins d'Enfant, and their monodromy groups. We did so for up to degree $N = 5$ in the hopes of generating an algorithm to generate Dessins from monodromy triples.

Process

1. Our first step towards creating a database was to create our own Dessins from the Monodromy Triples and Degree Sequences provided to us.
2. The next step was to record what we discovered electronically through LaTeX and TikZ.
3. Then, using a code on Mathematica we had to check that the Belyĭ maps generated our Dessins on the complex plane.
4. Although we ran into formatting issues we were able to put the elements of our database for g = 0 up to N = 5.

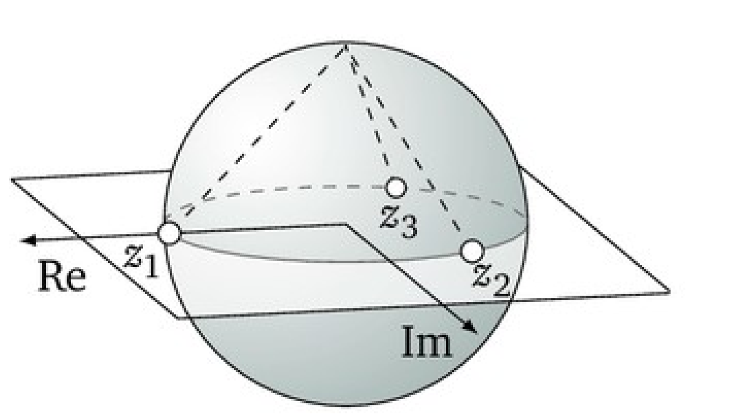
Background

- **Critical Values:** Consider a function $\beta : S \to \mathbb{P}^1(\mathbb{C})$ for the Riemann Sphere $S = \mathbb{P}^1(\mathbb{C})$. A critical point $P \in S$ satisfies $\beta'(P) = 0$. A critical value $w \in \mathbb{P}^1(\mathbb{C})$ is $x = \beta(P)$ the value of a critical point $P \in S$.
- **Belyĭ Maps:** Gennadiĭ Belyĭ [2] that a compact connected Riemann surface S of genus g is completely determined by the existence of a rational map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ which has three critical values. We say that a Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ is a rational map with critical values $\{0,\,1,\,\infty\}$.
- **Dessin d'Enfant:** Following an idea from Alexander Grothendieck [3], we define a Dessin d'Enfant (French for "child's drawing") as a bipartite graph with "black" vertices $B = \beta^{-1}(0)$, "white" vertices $W = \beta^{-1}(1)$, midpoints of faces $F = \beta^{-1}(\infty)$, and edges $E = \beta^{-1}\big([0,1]\big)$. For our purposes, these Dessins d'Enfant are projected onto the sphere using stereographic projection.
- **Degree Sequences:** For any $P \in B \cup W \cup F$, we will denote the ramification index e_P as the number of edges at vertex P. The collection of the ramification indices can be collected into a multiset of multisets called the degree sequence $\mathcal{D}$.

Stereographic Projection

$$\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \quad \to \quad S^2(\mathbb{R}) = \left\{ (u,v,w) \in \mathbb{A}^3(\mathbb{R}) \;\middle|\; u^2 + v^2 + w^2 = 1 \right\}$$

$$z = \frac{u + i\,v}{1 - w} \quad \mapsto (u,v,w) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right)$$



Belyĭ Maps → Permutation Triples

We can compute the Permutation Triples $(\sigma_0,\,\sigma_1,\,\sigma_\infty)$ from a given Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ with the following steps:

#1. Choose $x_0 \neq 0,\,1,\,\infty$; and compute the inverse image $\beta^{-1}(x_0) = \{P_1,\,P_2,\,\ldots,\,P_N\}$.

#2. Choose loops γ around $\epsilon = 0,\,1$ in $\mathbb{P}^1(\mathbb{C})$ that start and end at x_0. For example, we often choose $\gamma_\epsilon(t) = \epsilon + (x_0 - \epsilon)\,e^{2\pi i t}$.

#3. For each P_k, compute those paths $\widetilde{\gamma}_\epsilon^{(k)}$ on the Riemann Surface S such that $\beta \circ \widetilde{\gamma}_\epsilon^{(k)} = \gamma_\epsilon$ and $\widetilde{\gamma}_\epsilon^{(k)}(0) = P_k$.

#4. Compute permutations $\sigma_0,\,\sigma_1,\,\sigma_\infty \in S_N$ satisfying $\widetilde{\gamma}_\epsilon^{(k)}(1) = P_{\sigma_\epsilon(k)}$ and $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$

Permutation Triples → Degree Sequences

Adolf Hurwitz [4] showed the following properties for a Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ of degree N on a Riemann Surface S of genus g:

i. The composition $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$ is the trivial permutation, and the subgroup $\mathrm{Mon}(\beta) = \langle \sigma_0,\,\sigma_1,\,\sigma_\infty \rangle$ of the symmetric group S_N generated by them is a transitive subgroup. This is called the monodromy group of β.

ii. Each of these permutations is a product of disjoint cycles:

$$\begin{aligned}
\sigma_0 &= \prod_{P \in B} \left(b_{P,1}\; b_{P,2}\; \cdots\; b_{P,e_P} \right) & & & B &= \beta^{-1}(0) \\
\sigma_1 &= \prod_{P \in W} \left(w_{P,1}\; w_{P,2}\; \cdots\; w_{P,e_P} \right) & &\text{where} & W &= \beta^{-1}(1) \\
\sigma_\infty &= \prod_{P \in F} \left(f_{P,1}\; f_{P,2}\; \cdots\; f_{P,e_P} \right) & & & F &= \beta^{-1}(\infty)
\end{aligned}$$

iii. The multiset $\mathcal{D} = \big\{ \{e_P \,|\, P \in B\},\; \{e_P \,|\, P \in W\},\; \{e_P \,|\, P \in F\} \big\}$ is a collection of positive integers such that

$$N = \sum_{P \in B} e_P = \sum_{P \in W} e_P = \sum_{P \in F} e_P = |B| + |W| + |F| + (2\,g - 2).$$

iv. Conversely, any multiset $\mathcal{D}$ which a collection of three multisets is the degree sequence for some Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ if and only if there exist permutations $\sigma_0,\,\sigma_1,\,\sigma_\infty \in S_N$ such that the first three properties above hold.

There is a one-to-one correspondence between Belyĭ Maps, Dessins d'Enfant, and Permutation Triples – but there may be more than one of these for each Degree Sequence!

Elements of the Database

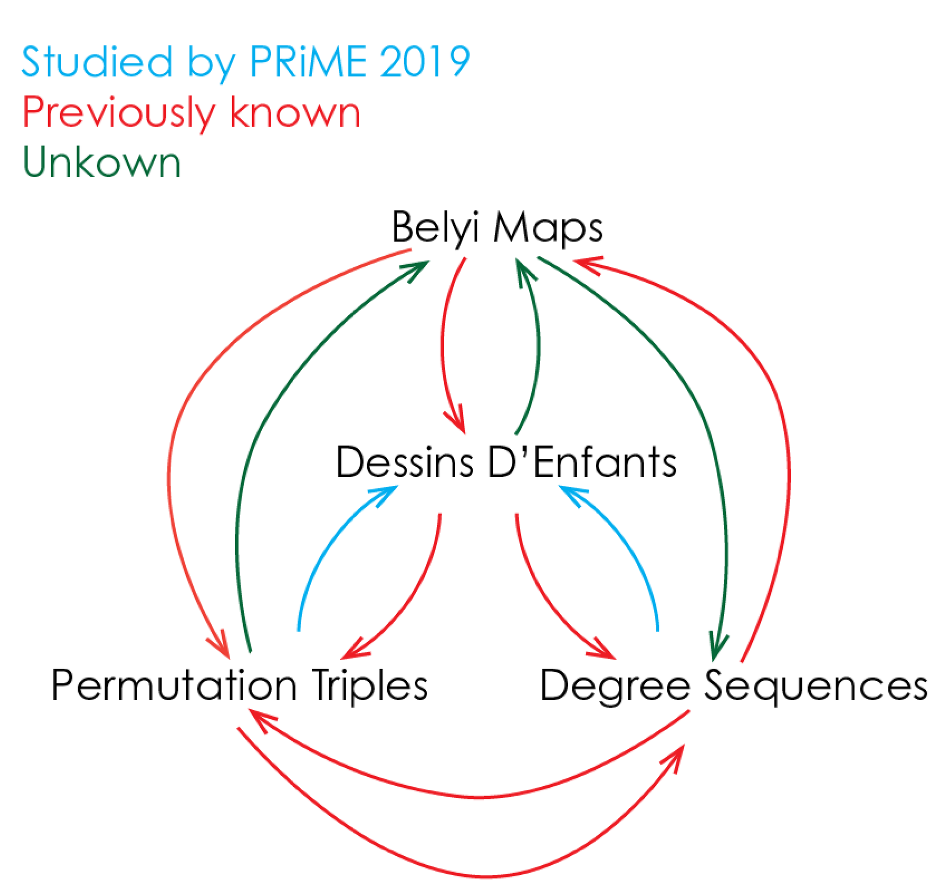


Figure 1: Elements of the Database

Motivating Question

How much of the database can we fill in if we don't have the Belyĭ Map?

Permutation Triples → Dessins

- For each disjoint cycle in σ_0 assign a black vertex with $|\,\sigma_0\,|$ edges coming out of it
  - Using the associated disjoint cycle, label the edges counterclockwise
- For each disjoint cycle in σ_1 assign a white vertex with $|\,\sigma_1\,|$ edges coming out of it
  - Using the associated disjoint cycle, label the edges counterclockwise
- Connect two vertices if they have an edge in common

Examples

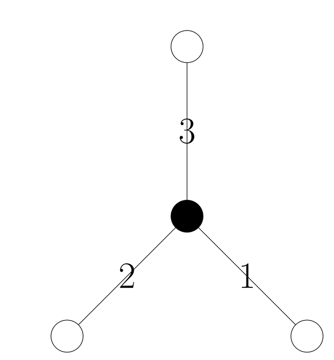


Figure 2: Dessin of **Degree** $N = 3$ & **Degree Sequence** $\mathcal{D} = \{\{3\},\,\{1,1,1\},\,\{3\}\}$

Permutation Triples: $\sigma_0 = (1\;3\;2)$, $\sigma_1 = (1)\,(2)\,(3)$, $\sigma_\infty = (1\;2\;3)$

Corresponding to $\beta = z^3$

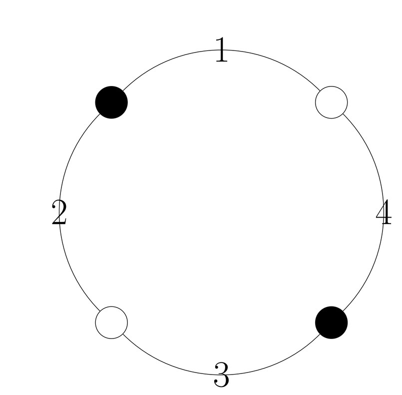
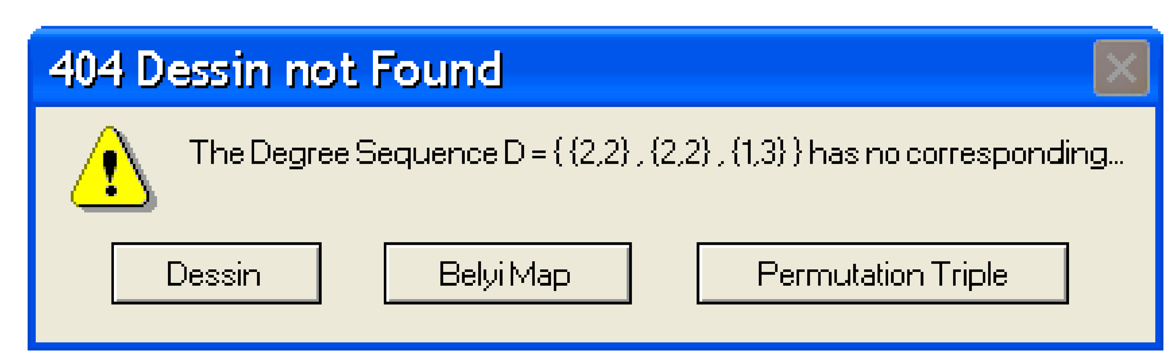


Figure 3: Dessin of **Degree** $N = 4$ & **Degree Sequence** $\mathcal{D} = \{\{2,2\},\,\{2,2\},\,\{2,2\}\}$

Permutation Triples: $\sigma_0 = (1\;2)\,(3\;4)$, $\sigma_1 = (1\;4)\,(2\;3)$, $\sigma_\infty = (1\;3)\,(2\;4)$

Corresponding to $\beta = \frac{(z^2+1)^2}{4z^2}$



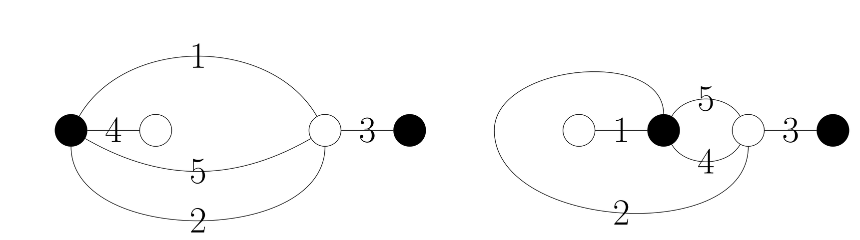


Figure 4: Dessins of **Degree** $N = 5$ & **Degree Sequence** $\mathcal{D} = \{\{4,1\},\,\{4,1\},\,\{2,2,1\}\}$

left

Permutation Triples: $\sigma_0 = (3)\,(1\;2\;5\;4)$, $\sigma_1 = (1\;5\;2\;3)\,(4)$, $\sigma_\infty = (5)\,(2\;3)\,(4\;1)$

Corresponding to:

$$\beta(z) = \frac{1}{4107}\frac{(-136i+4623)x^5+(-15096i+513153)x^4}{x^5+(40i+55)x^4+(13236i-12048)x^3+(1006992i-709956)x^2+(-67346586i-36186777)x-7475471046i-4016732247}$$

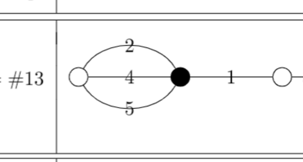
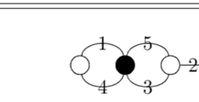
right

Permutation Triples: $\sigma_0 = (3)\,(1\;4\;5\;2)$, $\sigma_1 = (1)\,(5\;4\;2\;4)$, $\sigma_\infty = (2\;3)\,(4\;1)\,(5)$

Corresponding to:

$$\beta(z) = \frac{1}{4107}\frac{(136i+4623)x^5+(15096i+513153)x^4}{x^5+(-40i+55)x^4+(-13236i-12048)x^3+(-1006992i-709956)x^2+(67346586i-36186777)x+7475471046i-4016732247}$$

Database

Degree N	Genus g	Degree Sequence $\mathcal{D}$	Monodromy Triple	Belyĭ Map	Dessin d'Enfant
$N = 5$	$g = 0$	$\mathcal{D} = \{\{4,1\},\,\{3,2\},\,\{3,1,1\}\}$	$\sigma_0 = (1\;2\;4\;5)\,(3)$ $\sigma_1 = (1\;3)\,(2\;5\;4)$ $\sigma_\infty = (1\;2\;3)\,(4)\,(5)$	$\beta(z) = \#13$	
$N = 5$	$g = 0$	$\mathcal{D} = \{\{4,1\},\,\{3,2\},\,\{3,1,1\}\}$	$\sigma_0 = (1\;4\;3\;5)\,(2)$ $\sigma_1 = (1\;4)\,(2\;5\;3)$ $\sigma_\infty = (1\;2\;3)\,(4)\,(5)$	$\beta(z) = \#14$	

13. $\beta(z) = \frac{(-5568\sqrt{6}+14338)x^5+(153\sqrt{6}-1448)x^4}{5634x^5+(-2583\sqrt{6}-9432)x^4+(-2304\sqrt{6}-5076)x^3+(6686\sqrt{6}+16344)x^2+(11912\sqrt{6}+29178)x-18283\sqrt{6}-44784}$
14. $\beta(z) = \frac{(-5568\sqrt{6}+14338)x^5+(153\sqrt{6}-1448)x^4}{5634x^5+(-2583\sqrt{6}-9432)x^4+(-2304\sqrt{6}-5076)x^3+(6686\sqrt{6}+16344)x^2+(11912\sqrt{6}+29178)x-18283\sqrt{6}-44784}$

Future Work

- Find more Belyĭ maps!
- Update Algorithm to generate Dessin from Permutation Triples
- Compute on higher genera, degree, and other Riemann surfaces
- Create more movies to explain the other aspects of our Database

References

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[2] G. V. Belyi. "On extensions of the maximal cyclotomic field having a given classical Galois group." *J. Reine Angew. Math.*, 341 (1983), 147-156.

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[4] Adolph Hurwitz. "Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten." *Mathematische Annalen*, 39(1):1–60, 1891.

[5] Michael Klug, Michael Musty, Sam Schiavone, and John Voight. "Numerical calculation of three-point branched covers of the projective line." `https://arxiv.org/abs/1311.2081v3`

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PRiME Time!

`https://youtu.be/zUfb8AfGmPQ`